■ Definition Absolute value

The distance from the origin to the point a on the real number line is called the absolute value of the number a and is written $|a|$.

Four facts about absoute value follow immediately from the definition.

■ Theorem:

For any real number a,

(1) $a \geq 0 \Longrightarrow |a| = a$

(2) $a \leq 0 \Longrightarrow |a| = -a$

(3) $|a| \geq 0$

(4) $|a| = 0 \Longleftrightarrow a = 0$

Facts 1-4 are true of absolute value, *because absolute value means a distance*. Facts 1-4 make perfect sense if you replace the symbol "$(|a|)$" with the phrase "the distance of a to zero".

Facts (1) and (2) tell us how to rewrite an expression without the absolute value bars. If a is zero or positive, then $|a|$ is a. If a is negative, then the absolute value of a is given by $-(-a)$. Note that when a is negative, $-(-a)$ is a positive number and is in fact the distance of $-a$ from the origin of the real number line.

It is common to be a little sloppy and refer to facts 1-4 by saying "by definition". We will say "$|-5|$ is 5, by definition" and "$|-3|$ is 3, by definition".

[EX 1] $|7| = 7$, this is by fact (1), but you will here people say "by definition".

[EX 2] $|-7| = 7$, this is by fact (2), but you will here people say "by definition".

[EX 3] $|x|$ is what? Well, all we know about x is that it is some number. We cannot assume it is a positive number, nor may we assume it is negative. So, we must account for both cases by using facts (1) and (2). Here's how we write that:

Case 1: Suppose $x \geq 0$. $|x| = x$.

Case 2: Suppose $x < 0$. $|x| = -x$.

[EX 4] $|x-7|$ is what? Well, all we know about $x-7$ is that it is some number. We cannot assume it is a positive number, nor may we assume it is negative. So, we must account for both cases by using facts (1) and (2). Here's how we write that:

Case 1: Suppose $x-7 \geq 0$. $|x-7| = x-7$.

Case 2: Suppose $x-7 < 0$. $|x-7| = -(x-7)$.

[EX 5] $|x^2+2x+1|$ is what?

Case 1: Suppose $x^2+2x+1 \geq 0$. $|x^2+2x+1| = x^2+2x+1$.

Case 2: Suppose $x^2+2x+1 < 0$. $|x^2+2x+1| = -(x^2+2x+1)$.

[EX 6] $\left|\sqrt{2x+5}\right|$ is what?

Case 1: Suppose $\sqrt{2x+5} \geq 0$. $\left|\sqrt{2x+5}\right| = \sqrt{2x+5}$.

Case 2: Wait. Do we even need to consider the case when $\sqrt{2x+5} < 0$? No. WHY?

[EX 7] $\left|\frac{3x+7}{x^3-2x^2+x-12}\right|$ is what?

Case 1: Suppose $\frac{3x+7}{x^3-2x^2+x-12} \geq 0$. $\left|\frac{3x+7}{x^3-2x^2+x-12}\right| = \frac{3x+7}{x^3-2x^2+x-12}$.

Case 2: Suppose $\frac{3x+7}{x^3-2x^2+x-12} < 0$. $\left|\frac{3x+7}{x^3-2x^2+x-12}\right| = -\left(\frac{3x+7}{x^3-2x^2+x-12}\right)$.

The point: unless you know that the item between the absolute value bars is negative or unless you know it is positive, you have two cases to consider when you remove the bars.

■ What is meant by $|x-5|$?

Answer: $|x-5|$ means the distance from x to 5 on the number line.

But, shouldn't this mean "the distance of $(x-5)$ from zero on the number line?"

Here is how we fix that apparent inconistency:

$|a| = |a-0|$ and this means the distance from a to zero. Similarly, $|a-5|$ means the distance from a to 5.